Definition: Let $(R,\mathfrak{m})$ be a Noeth. one-dim domain.

For any R-module M,

$$h(M) := \min\{\lambda(R/J) \mid M \to J \to 0,\ J \subseteq R\}$$

$\lambda()$: length of a module

$$f \in \underset{\displaystyle \underline{M^*}}{\underset{\parallel}{\mathrm{Hom}_R(M,R)}} \quad J := f(M). \text{ Then look at } \lambda(R/J).$$

****** trace ideals: $M \in \text{Mod-}R$.

$$tr_R(M) := \sum_{\alpha \in M^*} \alpha(M)$$

For any such J in the above definition, it satisfies

$J \subseteq tr(M)$ clearly.

Thus, $\lambda(R/J) \geq \lambda(R/tr(M))$ for any such J

$$\Rightarrow \boxed{h(M) \geq \lambda(R/tr_R(M))}$$

· In fact, if M is rank 1, then $tr_R(M) = tr_R(J)$ $\forall$ such J.

$$* \left[\begin{array}{c} 0 \to \underset{\tau(M)}{(\ \ )} \to M \to J \to 0 \\ \mathrm{Hom}_R(\tau(M),R) = 0 \end{array}\right].$$

Observation: Suppose $\boxed{h(M) = \lambda(R/J)}$

Then $\boxed{h(J) = \lambda(R/J).}$ ← Clear

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr] & J \arrow[r] \arrow[d, "\wr"', "g"] & 0 \\
 & I \arrow[dr] & \\
 & & 0
\end{tikzcd}$$

So enough to just study $h(J)$ for ideals J.

So, enough to just study $\underline{h(J)}$. for ideals J.

- What does isomorphism of ideals mean?

$f : I \longrightarrow J$. Take $x \in I$, $y \in I$.

$$yf(x) = f(xy) = xf(y) \Rightarrow \frac{f(x)}{x} = \frac{f(y)}{y} \quad \forall \underset{(\neq 0)}{x, y} \in I.$$

$$f(r) = \frac{f(r)}{r} \cdot r = \frac{f(x)}{x} \cdot r$$

So, any such f is just multiplication by an element $\alpha \in K = \mathrm{Frac}(R)$.

$$\boxed{I \simeq J \iff I = \frac{a}{b} J \iff bI = aJ}$$

Assumptions throughout:

Let (R, m, k) be st.
- $\dim R = 1$
- R is a domain
- $\mathrm{Frac}(R) = K$
- $\overline{R}$ integral closure in K
- $\hat{R}$ is reduced.
- $\overline{R}$ is a DVR

Examples: $R = k[[t^3, t^4, t^5]]$

$R = k[[t^3 + t^{71}, t^{109}, t^{\cdot\cdot}]]$

$\overline{R} = k[[t]]$

Fact: (Corollary 4.6.2 Huneke-Swanson)
Under the above assumptions, $\overline{R}$ is a finite R-module of rank 1. (Same fraction field)
- birational extension

Lemma For any $\overset{\neq 0}{x} \in R$,

$$\boxed{\lambda(R/xR) = \lambda(\overline{R}/x\overline{R})}$$

Pf R & $\overline{R}$ are both Maximal Cohen Macaulay over R.

$$\lambda(\overline{R}/x\overline{R}) = \underset{\text{(Hilbert-Samuel multiplicity)}}{e(x;\overline{R})} = e(x;R)\,\operatorname{rank}_R \overline{R} = e(x;R) = \lambda(R/xR).$$

$\square$

Theorem $h(J) = \lambda(R/J) \iff \underline{R:_K J} \subseteq \overline{R}$ $\quad [R:_K J := \{\alpha \in K / \alpha J \subseteq R\}]$

$\overset{\Vert}{} \cdot\ J^{-1}, J^*$ $\quad \left[\begin{array}{l} (J:_K J) \subseteq \overline{R} \quad \text{det-trick.} \\ \alpha J \subseteq J \Rightarrow \alpha \in \overline{R} \end{array}\right]$

Pf $\boxed{\Leftarrow}$ $\boxed{R:_K J \subseteq \overline{R}.}$

Suppose if possible, $h(J) = \underline{\lambda(R/I) < \lambda(R/J)}$ and $I \simeq J$.

$I = \frac{a}{b}J. \Rightarrow \boxed{bI = aJ}$ $\qquad \frac{a}{b}J = I \subseteq R.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \frac{a}{b} \in \underline{R:_K J}.$

By assumption, $\frac{a}{b} \in \overline{R}$. $\qquad \Rightarrow \frac{a}{b} \in \overline{R}$

$$\Rightarrow \boxed{a\overline{R} \subseteq b\overline{R}}$$

$$\lambda(\overline{R}/a\overline{R}) \geqslant \lambda(\overline{R}/b\overline{R})$$

$aJ \subseteq aR \subseteq R.$

$$0 \to \frac{aR}{aJ} \to \frac{R}{aJ} \to R/aR \to 0$$

$bI \subseteq bR \subseteq R$

$$0 \to R/I \to \frac{R}{bI} \to R/bR \to 0$$

$$\lambda(R/J) + \lambda(R/aR) = \lambda(R/aJ)$$

$$\overset{\checkmark}{\lambda}(R/I) + \lambda(R/bR) = \overset{\Vert}{\lambda(R/bI)}$$

$\cdot\ \lambda(R/I) < \lambda(R/J) \Rightarrow \lambda(R/bR) > \lambda(R/aR)$

$$\overset{\text{Lemma}}{\Longrightarrow} \lambda(\overline{R}/b\overline{R}) > \lambda(\overline{R}/a\overline{R}) \to \leftarrow.$$

$$\boxed{h(J) = \lambda(R/J).}$$ ✓

$\boxed{\Rightarrow}$ $h(J) = \lambda(R/J)$. N.T.S. $R:_K J \subseteq \overline{R}$.

$\boxed{\Leftarrow}$ $h(J)=\lambda(R/J)$. N.T.S. $K:_K \vee \subseteq K$.

Take $\frac{a}{b} \in R:_K J$. Assume $\frac{a}{b} \notin \overline{R}$. $\Rightarrow$ $a\overline{R} \notin b\overline{R}$.

But $\overline{R}$ is a DVR. Thus, $\boxed{b\overline{R} \subsetneq a\overline{R}}$

Set $I := \frac{a}{b} J$ $I \simeq J$ & $bI = aJ$

$$\begin{array}{ccccc} \lambda(R/I) & + & \lambda(R/bR) & = & \lambda(R/bI) \\ \rotatebox{90}{$\leq$} & & & & \| \\ \lambda(R/J) & + & \lambda(R/aR) & = & \lambda(R/aJ) \end{array} \qquad \left[\begin{array}{l} h(J)=\lambda(R/J). \\ I \simeq J. \Rightarrow \lambda(R/I) \geq \lambda(R/J) \end{array}\right]$$

$\Rightarrow$ $\lambda(R/aR) \geq \lambda(R/bR)$

$\boxed{\text{Lemma}}$ $\Rightarrow$ $\lambda(\overline{R}/a\overline{R}) \geq \lambda(\overline{R}/b\overline{R}) > \lambda\left(\frac{\overline{R}}{a\overline{R}}\right) \rightarrow\leftarrow$.

Thus, $h(J)=\lambda(R/J) \Leftrightarrow R:_K J \subseteq \overline{R}$.

$\square$

$\boxed{\text{Conductor}}$

$\boxed{\mathfrak{C} := R:_K \overline{R}}$, largest common ideal of R and $\overline{R}$.

i.e. $\mathfrak{C} = \mathfrak{C}\overline{R}$.

$R:_K(R:_K I) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(I,R),R)$

I^{**}

$\boxed{\text{Claim:}}$ $\boxed{\overline{R} = R:_K \mathfrak{C}}$

$\boxed{\text{Pf}}$ By defn: $\mathfrak{C}\overline{R} \subseteq R$ & hence, $\overline{R} \subseteq R:_K \mathfrak{C}$. ✓

$\underline{R:_K \mathfrak{C}} = R:_K(\mathfrak{C}\overline{R}) = (R:_K \overline{R}):_K \mathfrak{C} = \boxed{\mathfrak{C}:_K \mathfrak{C}} \subseteq \overline{R}$. ✓

$\boxed{h(\mathfrak{C}) = \lambda(R/\mathfrak{C})}$ $R:_K \mathfrak{C} = \overline{R}$

$\boxed{☆}$ $\boxed{\text{Remark:}}$ $I:_K I = R:_K I$ is equivalent to being a trace ideal (Look at Haydee Lindo; Goto Shiro, ...)

$I:_K I \subseteq \overline{R}$

So, for any trace ideal $\boxed{h(I) = \lambda(R/I)}$

?

$\boxed{\text{Rmk:}}$ Herzog-Hibi-Stamate $\mathfrak{C} \subseteq tr(I)$ for any I.

Rmk: Herzog-Hibi-Stamate ? $\mathfrak{C} \subseteq \operatorname{tr}(I)$ for any I.

- $\boxed{\text{Take any } J \supseteq \mathfrak{C}}$. $\Rightarrow$ $R :_K J \subseteq R :_K \mathfrak{C} = \overline{R}$.

$$\Rightarrow h(J) = \lambda(R/J).$$

Take any rank 1-module.

$$\boxed{0 \to \overset{0}{\tau(M)} \to \overset{1}{M} \to \overset{1}{J} \to 0}$$

1 ranks.

torsion

$M \to J \to 0$

$h(M)$.

$\tau(\Omega_{R/k})$ $\quad \Omega_{R/k}$

R is regular $\Leftrightarrow$ $\tau(\Omega_{R/k}) = 0$.

Thm: (Prashanth, Hailong) For any ideal I, $I^{**} \simeq J$ where $\mathfrak{C} \subseteq J$.

Pf Take I. Let J_1 be s.t.

$$\boxed{I \simeq J_1} \text{ and } h(I) = \boxed{\lambda(R/J_1) = h(J_1)}$$

Hence, $R :_K J_1 \subseteq \overline{R}$. by prv Thm:

$f : \mathbb{I} \to \mathbb{R}$

$$\Rightarrow \boxed{R :_K (R :_K J_1)} \supseteq R :_K \overline{R} = \mathfrak{C}$$

$\| \quad J.$

$J_1^{**} \simeq I^{**}$.

$J \supseteq \mathfrak{C}$.

$I^{**} \simeq J \supseteq \mathfrak{C}$. $\square$

Thus, in particular, to study isomorphism classes of reflexive ideals, it is enough to look at reflexive ideals containing the conductor

$\boxed{\operatorname{Ref}(R)}$ $\quad \boxed{\operatorname{CM}(R)}$ Graham Leuschke's.

finite type. ↳ finite type := there are finitely many such indecomposable objects upto isomorphism

↓ Ref (R)

$\downarrow$

$\mathrm{Ref}_1(R)$ indecomposable "objects" upto isomorphism

"class of reflexive ideals".

for example, if $\lambda(R/I) \leq 7$,

$\boxed{\mathrm{Ref}_1(R) \text{ is indeed of finite type}}$.

$\boxed{I = m}$: 1-step normal ; Eleanore Faber.